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Summing the instanton series in $\mathcal{N}=2$ superconformal large-N QCD

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Abstract: We consider the multi-instanton collective coordinate integration measure in $\mathcal{N}=2$ supersymmetric SU(N) gauge theory with N_F fundamental hypermultiplets. In the large-N limit, at the superconformal point where $N_F=2N$ and all VEVs are turned off, the k-instanton moduli space collapses to a single copy of $AdS_5\times S^1$. The resulting k-instanton effective measure is proportional to $\sqrt{N}g^4\hat{\mathcal{Z}}_k^{(6)}$, where $\hat{\mathcal{Z}}_k^{(6)}$ is the partition function of $\mathcal{N}=(1,0)$ SYM theory in six dimensions reduced to zero dimensions. The multi-instanton can in fact be summed in closed form. As a hint of an AdS/CFT duality, with the usual relation between the gauge theory and string theory parameters, this precisely matches the normalization of the charge-k D-instanton measure in type IIB string theory compactified to six dimensions on $K3$ with a vanishing two-cycle.

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Recently, it has been shown that the multi-instanton measure of $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theory, in the conformal case of vanishing VEVs, at large N, has a precise correspondence with superstring theory on $AdS_5 \times S^5$ in the following sense [1, 2]:

(i) The effective large-N collective coordinate space ('moduli space') of the charge k instanton sector is $AdS_5 \times S^5$.

(ii) The large-N collective coordinate integration measure for k instantons matches precisely the measure for D-instantons in the string theory; in particular, it is proportional to $\sqrt{N} g^8 \hat{\mathcal{Z}}_k^{(10)}$, where $\hat{\mathcal{Z}}_k^{(10)}$ is the partition function for $\mathcal{N}=(1,0)$ supersymmetric Yang-Mills (SYM) theory in ten dimensions reduced to zero dimensions.

(iii) Various correlation functions which have instanton contributions match those calculated from string corrections to the type IIB supergravity effective action as required by the AdS/CFT correspondence [3]–[8].

These features add up to compelling circumstantial evidence in favor of Maldacena's conjecture.

It is now tempting to apply the same large-N instanton techniques to theories without an obvious string theory dual in the hope that the Yang-Mills instanton calculation will give some clues about the latter. From a Yang-Mills perspective a simple variation on the $\mathcal{N}=4$ theory is $\mathcal{N}=2$ supersymmetric SU(N) gauge theory with N_F fundamental hypermultiplets. As in the $\mathcal{N}=4$ case, we consider the theory with vanishing VEVs. For $N_F=2N$ the theory then has non-trivial superconformal invariance (just as the $\mathcal{N}=4$ theory); a priori we expect that this case might be the most similar to its $\mathcal{N}=4$ cousin because the coupling does not run and the instanton calculation can be justified for small values of the 't Hooft coupling g^2N.

In analogy to the $\mathcal{N}=4$ model, we shall find that the multi-instanton series may be summed in closed form, and the following features emerge:[1]

(i) The effective large-N moduli space of the charge k instanton sector is now $AdS_5 \times S^1$.

[1]Unlike the $\mathcal{N}=4$ case [2], here we will not study correlation functions per se since a "dictionary" relating gauge-invariant operators on the SYM and string sides of a proposed equivalence is missing in the present case. To leading semiclassical order, n-point correlators $\langle \Phi_1(x_1)\cdots\Phi_n(x_n)\rangle$ are straightforwardly obtained from the collective coordinate measure obtained below, by inserting $\Phi_1(x_1)\times\cdots\times\Phi_n(x_n)$ under the integral, where each Φ_i is replaced by the corresponding classical expression evaluated on the bosonic and/or fermionic collective coordinates of the multi-instanton.

(ii) The large-N collective coordinate integration measure for k instantons is proportional to $\sqrt{N}g^4\hat{\mathcal{Z}}_k^{(6)}$, where $\hat{\mathcal{Z}}_k^{(6)}$ is the partition function for $\mathcal{N}=(1,0)$ SYM theory in six dimensions reduced to zero dimensions. While an AdS/CFT correspondence is not presently known for this $\mathcal{N}=2$ model, it is suggestive that this expression precisely matches the normalization of the charge-k D-instanton measure in type IIB string theory compactified to six dimensions on $K3$ with a vanishing two-cycle [9].

In performing the $\mathcal{N}=2$ calculation, we shall find that many of the required formulae and notation can be adapted in an obvious way from the $\mathcal{N}=4$ theory, discussed at great length in [2]. For this reason we shall include only a paucity of details and refer frequently to this latter reference. In the $\mathcal{N}=4$ theory the fermion and scalar fields come with $\mathrm{SU}(4)_R$ indices $A,B,\ldots=1,2,3,4$. For the present theory, in order to describe the fermions and scalars of the vector multiplet we only let $A,B,\ldots=1,2$. The scalar field is then the single complex field $A^{12}=-A^{21}$ and the gluinos come in a pair λ^A, $A=1,2$, which rotate into one another under $\mathrm{SU}(2)_R$. Virtually all the formulae of [2] for these fields in the background of an instanton may be carried over. The new ingredients are the fundamental matter fields. These, too, have been discussed at great length, in [10], and we shall borrow profusely from this reference also.

The charge k instanton solution is described by the following bosonic collective coordinates: the $k\times k$ dimensional hermitian matrices a'_n, $n=1,2,3,4$; the $N\times k$ dimensional matrices $w_{\dot\alpha}$, where $\dot\alpha=1,2$ is a Weyl index; and their conjugates $\bar w^{\dot\alpha}$. These are subject to quadratic ADHM constraints reviewed below. The gluino zero modes, λ^A, $A=1,2$, in the background of the multi-instanton are described by the following Grassmann collective coordinates: the $k\times k$ dimensional hermitian matrices $\mathcal{M}'^A_\alpha$, where $\alpha=1,2$; the $N\times k$ dimensional matrices μ^A; and their conjugates $\bar\mu^A$. These coordinates are subject to fermionic ADHM constraints which are likewise reviewed below. Sometimes it is convenient to assemble separately the bosonic and fermionic collective coordinates into $(N+2k)\times k$ dimensional matrices $a=\left(\begin{smallmatrix} w_{\dot\alpha}\\ a'_{\alpha\dot\alpha}\end{smallmatrix}\right)$ and $\mathcal{M}^A=\left(\begin{smallmatrix}\mu^A\\ \mathcal{M}'^A_\alpha\end{smallmatrix}\right)$, where $a'_{\alpha\dot\alpha}=a'_n\sigma^n_{\alpha\dot\alpha}$.

In addition to the collective coordinates coming from the vector multiplet, there are additional fermionic (but not bosonic) collective coordinates for the fundamental matter fields denoted $\mathcal{K}$ and $\tilde{\mathcal{K}}$, which are $k\times N_F$ and $N_F\times k$ dimensional matrices, respectively, of Grassmann numbers [10]. These collective coordinates account for the $2kN_F$ fundamental fermion zero modes in the k instanton background. Unlike the collective coordinates describing the vector multiplet there are no ADHM constraints on $\mathcal{K}$ and $\tilde{\mathcal{K}}$.

Of vital importance in constructing the instanton measure is the action of the theory evaluated on the instanton solution, with all the fermion modes 'turned on'. In the case when the VEVs vanish, the action of the theory evaluated on the multi-instanton solution has the form [10]

$$S^k_{\text{inst}} = \frac{8\pi^2 k}{g^2} - ik\theta + S^k_{\text{quad}}\,, \tag{1}$$

where the term S^k_{quad} is a particular fermion quadrilinear term of the form

$$S^k_{\text{quad}} = -\frac{8\pi^2}{g^2}\,\mathrm{tr}_k\,\Lambda_{\text{hyp}}\mathbf{L}^{-1}\Lambda_{\text{vec}}\,. \tag{2}$$

Here we have defined the $k \times k$ matrix fermion bilinears

$$\begin{aligned}\Lambda_{\text{vec}} &= \frac{1}{2\sqrt{2}}\left(\bar{\mathcal{M}}^1\mathcal{M}^2 - \bar{\mathcal{M}}^2\mathcal{M}^1\right),\\ \Lambda_{\text{hyp}} &= \frac{\sqrt{2}}{8}\mathcal{K}\tilde{\mathcal{K}}\end{aligned} \tag{3}$$

and $\mathbf{L}$ is linear operator on $k \times k$ matrices defined in [10, 2]. The existence of a four-fermion interaction (2) between the fermionic collective coordinates played a vital rôle in the $\mathcal{N} = 4$ theory and we expect it to do so here as well.

The four-fermion interaction actually lifts all the fermion zero modes except the 8 supersymmetric and superconformal adjoint zero modes ζ^A_α and $\bar{\eta}^{\dot{\alpha}A}$ which will be explicitly defined later (or see [2]). From the form of the quadrilinear action (2), we see that adjoint zero modes, numbering $4kN$, and fundamental zero modes, numbering $2kN_F$, are lifted in equal number. In any given n-point function, the n insertions must, of course, saturate the integrals over the 8 exact supersymmetric and superconformal adjoint zero modes but furthermore let us suppose that they saturate the integrals over $n_a - 8$ extra adjoint fermion zero modes and n_f fundamental fermion zero modes. We then have the following selection rule:

$$n_a - n_f = 2k(2N - N_F)\,. \tag{4}$$

From this selection rule it immediately follows that:

(i) Unless one requires $N_F \sim 2N$ in the large-N limit, the only non-vanishing correlators in this limit are ones where the number of insertions grows with N. In this case the insertions will necessarily modify the large-N saddle-point equations derived below (see for example the analysis of ref. [11]).

(ii) It is obvious from eq. (4) that it is only in the conformal case $N_F = 2N$ that a given correlation function gets a contribution from each instanton number k rather than from only one value of k. Moreover, in the absence of VEVs, only in the conformal case can one choose an arbitrarily small coupling constant and hence rigorously justify the instanton approximation (again, see ref. [11] for an analysis of what goes wrong with instanton physics when the VEVs are zero but the β-function is not).

For these reasons, we concentrate henceforth on the conformal case $N_F = 2N$.

The next ingredient needed in the analysis is the integration measure $d\mu^k_{\rm vec}$ on the space of collective coordinates of the vector multiplet. This has been determined in refs. [12, 13, 10, 2]. It is simply the 'flat cartesian measure' for the bosonic and fermionic collective coordinates with explicit delta functions that impose the aforementioned bosonic and fermionic ADHM constraints:[2]

$$\int d\mu^k_{\rm vec} = \frac{2^{-k^2/2}}{{\rm Vol}\, U(k)} \int d^{4k^2}a'\, d^{2kN}\bar w\, d^{2kN}w \prod_{A=1,2} d^{2k^2}\mathcal{M}'^A d^{kN}\bar\mu^A d^{kN}\mu^A \times$$
$$\times \prod_{r=1,\dots,k^2}\Bigg[\prod_{c=1,2,3}\delta\left(\frac{1}{2}{\rm tr}_k\, T^r({\rm tr}_2\ \tau^c\bar a a)\right)\times$$
$$\times \prod_{A=1,2}\prod_{\dot\alpha=1,2}\delta\left({\rm tr}_k\, T^r(\bar{\mathcal{M}}^A a_{\dot\alpha} + \bar a_{\dot\alpha}\mathcal{M}^A)\right)\Bigg]({\rm det}_{k^2}\mathbf{L})^{-1}\,. \quad (5)$$

The full measure is given by

$$\int d\mu^k_{\rm phys} = (C_1')^k \int d\mu^k_{\rm vec}\times d\mu^k_{\rm hyp}\,, \quad (6)$$

where $d\mu^k_{\rm hyp}$ is the measure for the fermionic collective coordinates from the matter fields; this is simply

$$\int d\mu^k_{\rm hyp} = \pi^{-2kN_F}\int d^{kN_F}\mathcal{K}\, d^{kN_F}\tilde{\mathcal{K}}\,, \quad (7)$$

where the integrals $d^{kN_F}\mathcal{K}$ and $d^{kN_F}\tilde{\mathcal{K}}$ are defined as separate integrals over each component of the matrices $\mathcal{K}$ and $\tilde{\mathcal{K}}$ [10]. In (6) the constant C_1' is determined by a detailed comparison of this measure with the one instanton measure of Bernard for SU(N) [14] suitably generalized to an $\mathcal{N}=2$ theory with fundamental hypermultiplets. This comparison gives

$$C_1' = 2^{1/2}\pi^{-2N}\Lambda^{2N-N_F}\,, \quad (8)$$

where Λ is the dynamical mass scale in the Pauli-Villars regularization scheme.

Following the same strategy as in [10, 2], we can bilinearize the four fermion interaction by introducing a set of auxiliary variables χ_a, $a=1,2$, a two-vector of hermitian $k\times k$ matrices. In addition, it is convenient to define the matrix of complex variables

$$\chi = \chi_1 + i\chi_2\,. \quad (9)$$

[2] The integrals over $k\times k$ matrix quantities a_n' and $\mathcal{M}'^A_\alpha$ and the arguments of the delta functions are defined with respect to a basis of hermitian matrices T^r, $r=1,\dots,k^2$, normalized so that ${\rm tr}_k\, T^rT^s=\delta^{rs}$.

The transformation we need was established in [10] and later adapted to the $\mathcal{N}=4$ theory in [2]:

$$(\det_{k^2}\mathbf{L})^{-1}\exp -S^k_{\rm quad} = \pi^{-k^2}\int d^{2k^2}\chi\,\exp\Big[-{\rm tr}_k\,\chi_a\mathbf{L}\chi_a + \sqrt{8}\pi g^{-1}{\rm tr}_k\,\Lambda_{\rm hyp}\chi + \\ + \sqrt{8}\pi g^{-1}{\rm tr}_k\,\chi^\dagger\Lambda_{\rm vec}\Big]\,. \qquad (10)$$

Conveniently, the factor of $(\det_{k^2}\mathbf{L})^{-1}$ needed in this gaussian rewriting is supplied by the flat measure, (5). Next we transform to the so-called gauge invariant measure [2]; this change of variables has the enormous calculational advantage that it allows us to integrate out explicitly both the bosonic and fermionic ADHM constraints. The necessary manipulations are detailed in [2]. The resulting measure involves the bilinear bosonic variables

$$W^{\dot\alpha}_{\ \dot\beta} = \bar w^{\dot\alpha}w_{\dot\beta}\,,\qquad W^0 = {\rm tr}_2\,W\,,\qquad W^c = {\rm tr}_2\,\tau^cW\,,\qquad c=1,2,3\,, \qquad (11)$$

where W^0 and W^c are $k\times k$ hermitian matrices, and a decomposition of the fermionic collective coordinates $\{\mu^A,\bar\mu^A\}$:

$$\mu^A = w_{\dot\alpha}\zeta^{\dot\alpha A} + \nu^A\,,\qquad \bar\mu^A = \bar\zeta^A_{\dot\alpha}\bar w^{\dot\alpha} + \bar\nu^A\,, \qquad (12)$$

where $\{\nu^A,\bar\nu^A\}$ lie in the orthogonal subspace to w in the sense that

$$\bar w^{\dot\alpha}\nu^A = 0\,,\qquad \bar\nu^A w_{\dot\alpha} = 0\,. \qquad (13)$$

Here $\zeta^{\dot\alpha A}$ and $\bar\zeta^A_{\dot\alpha}$ are $k\times k$ matrices of Grassmann spinors. The bosonic coordinates W^c, $c=1,2,3$ and the fermionic coordinates $\bar\zeta^A_{\dot\alpha}$ are then explicitly integrated-out using, respectively, the delta-functions in (5) whose arguments are the bosonic and fermionic ADHM constraints.

As in the $\mathcal{N}=4$ model discussed in ref. [2], the next stage involves explicitly integrating out the ν^A's and $\bar\nu^A$'s. Concentrating on the final term in the exponent in eq. (10), one expands:

$$\Lambda_{\rm vec} = \frac{1}{\sqrt8}\left(\bar\nu^1\nu^2 - \bar\nu^2\nu^1\right) + \tilde\Lambda_{\rm vec}\,. \qquad (14)$$

Here we have defined a $k\times k$ matrix fermion bilinear in the remaining — unintegrated — fermionic collective coordinates:

$$\tilde\Lambda_{\rm vec} = \frac{1}{2\sqrt2}\left(\bar\zeta^1_{\dot\alpha}W^{\dot\alpha}_{\ \dot\beta}\zeta^{\dot\beta2} - \bar\zeta^2_{\dot\alpha}W^{\dot\alpha}_{\ \dot\beta}\zeta^{\dot\beta1} + \{\mathcal{M}'^{\alpha1},\mathcal{M}'^2_\alpha\}\right)\,, \qquad (15)$$

where $\bar\zeta^A_{\dot\alpha}$, in turn, is defined in terms of $\zeta^{\dot\alpha A}$ and $\mathcal{M}^A_\alpha$ using the fermionic ADHM constraints. Performing the integration over the ν^A's and $\bar\nu^A$'s then gives:

$$\int\prod_{A=1,2}d^{k(N-2k)}\nu^A\,d^{k(N-2k)}\bar\nu^A\,\exp\left[\pi g^{-1}{\rm tr}_k\,\chi^\dagger(\bar\nu^1\nu^2-\bar\nu^2\nu^1)\right] = \\ = \left(\frac{\pi}{g}\right)^{2k(N-2k)}\left(\det_k\chi^\dagger\right)^{2(N-2k)}. \qquad (16)$$

This term will contribute to the saddle-point at large N.

Next we turn to the fermionic collective coordinates in the matter sector. Focusing on the middle term in the exponent in eq. (10) and integrating out the $\mathcal{K}$ and $\tilde{\mathcal{K}}$ modes gives (for general N_F):

$$\int d^{kN_F}\mathcal{K}\, d^{kN_F}\tilde{\mathcal{K}}\exp\left[\frac{1}{2}\pi g^{-1}\mathrm{tr}_k\,\mathcal{K}\tilde{\mathcal{K}}\chi\right] = \left(\frac{\pi}{2g}\right)^{kN_F}\left(\det{}_k\chi\right)^{N_F}. \tag{17}$$

Putting together all the previous steps, we now have a form for the measure which is amenable to a large-N limit:

$$\begin{aligned}\int d\mu^k_{\rm phys}\, e^{-S^k_{\rm inst}} = {} & \frac{2^{5k^2/2-kN_F}\pi^{k(2N-N_F)-4k^2}(C_1')^k c_{k,N}}{g^{k(2N-N_F)-4k^2}\,\mathrm{Vol}\,U(k)}e^{2\pi ik\tau}\times\\ & \times\int d^{k^2}W^0\, d^{4k^2}a'\, d^{2k^2}\chi\prod_{A=1,2}d^{2k^2}\mathcal{M}'^A\, d^{2k^2}\zeta^A\times\\ & \times\left(\det{}_{2k}W\right)^{N-2k}\left(\det{}_k\chi^\dagger\right)^{2(N-2k)}\left(\det{}_k\chi\right)^{N_F}\times\\ & \times\exp\left[\sqrt{8}\pi g^{-1}\,\mathrm{tr}_k\,\chi^\dagger\tilde{\Lambda}_{\rm vec}-\mathrm{tr}_k\,\chi_a\mathbf{L}\chi_a\right].\end{aligned} \tag{18}$$

Here the constant $c_{k,N}$ comes from switching to gauge-invariant variables, and is given by [2]:

$$c_{k,N} = \frac{2^{2kN-4k^2+k}\pi^{2kN-2k^2+k}}{\prod_{i=1}^{2k}(N-i)!}. \tag{19}$$

As in [2], in order to facilitate taking the large-N limit, it is useful to re-scale χ:

$$\chi\to\sqrt{N}\chi. \tag{20}$$

With this rescaling, the last term in the exponent of eq. (18) now scales like N, hence contributes to the large-N saddle-point equations (as do the three determinantal factors preceding the exponential). Collecting all the bosonic terms which have the form of an exponent of "N times something" and specializing once again to $N_F = 2N$ defines the "large-N effective instanton action":

$$S_{\rm eff} = -\log\det{}_{2k}W - 2\log\det{}_k\chi^\dagger - 2\log\det{}_k\chi + \mathrm{tr}_k\,\chi_a\mathbf{L}\chi_a\,. \tag{21}$$

We now turn to the solution of the large-N saddle-point equations. These are the coupled Euler-Lagrange equations that come from extremizing $S_{\rm eff}$ with respect to the remaining bosonic collective coordinates in the problem. The analysis of these coupled equations, while rather involved, is virtually identical to the $\mathcal{N}=4$ model of ref. [2]; hence we suppress the calculational details. As in that model, the saddle-point solution that is dominant in the large-N limit is given by:

$$W^0 = 2\rho^2\,1_{[k]\times[k]}\,,\qquad \chi_a = \rho^{-1}\hat{\Omega}_a\,1_{[k]\times[k]}\,,\qquad a'_n = -X_n\,1_{[k]\times[k]}\,, \tag{22}$$

where $\hat{\Omega}_a$ is a unit 2-vector. It is convenient to parametrize $\hat{\Omega}_a$ by the phase angle ϕ:

$$\hat{\Omega}_1 + i\hat{\Omega}_2 = e^{i\phi}\,. \tag{23}$$

The interpretation of the solution is as in [2] with the S^5 replaced by S^1, parametrized by $\hat{\Omega}_a$. In other words, the solution is parametrized by a point (X_n, ρ, ϕ) on $AdS_5 \times S^1$. As in the $\mathcal{N} = 4$ case, the solution (22) is not actually the most general solution to the saddle-point equations; there are additional flat directions (moduli). However, we shall find in a completely analogous way that the integrals over the additional moduli are actually convergent; as such, they are more conveniently viewed as fluctuations around, rather than facets of, the maximally symmetric saddle-point solution (22).

The next stage of the analysis is to expand the effective action in the fluctuations out to sufficient order to ensure the convergence of the integrals over the fluctuations. Fortunately, the analysis need not be repeated because our effective action (21) is, up to a constant, simply the $\mathcal{N} = 4$ effective action of [2] with the replacements $\chi_{12} \to \chi/\sqrt{8}$, $\chi_{34} \to \chi^\dagger/\sqrt{8}$ and all other components of χ_{AB} set to zero. The fluctuations in W^0 are integrated out at gaussian order to leave integrals over the remaining fluctuations in $\hat{a}'_n$ and $\hat{\chi}_a$, the traceless parts, that are lifted at quartic order; the terms beyond quartic order are then formally suppressed by (fractional) powers of $1/N$, and may be dropped. The 'action' for these quartic fluctuations is precisely the action for six-dimensional Yang-Mills theory with gauge field

$$A_\mu = N^{1/4}\big(\rho^{-1}\hat{a}'_n, \rho\hat{\chi}_a\big)\,, \qquad \mu = 1, \ldots, 6\,, \tag{24}$$

dimensionally reduced to 0 dimensions, i.e. with all derivatives set to zero:

$$S(A_\mu) = NS_{\text{eff}} = -\frac{1}{2}\text{tr}_k\,[A_\mu, A_\nu]^2 + \cdots\,. \tag{25}$$

In (24) the hats on the variables imply that they are traceless $k \times k$ matrices, i.e. SU(k)-valued rather than $U(k)$.

Not surprisingly the coupling to the fermions completes the six-dimensional theory to an $\mathcal{N} = (1, 0)$ supersymmetric gauge theory in six dimensions dimensionally reduced to 0 dimensions. It is easy to see, for instance, that this theory has the requisite number of supersymmetries. The eight dimensional fermion field of the $\mathcal{N} = (1, 0)$ theory has components

$$\Psi = \sqrt{\frac{\pi}{2g}}\,N^{1/8}e^{-i\phi/2}\big(\rho^{-1/2}\hat{\mathcal{M}}'^{A}_{\alpha}\,,\ \rho^{1/2}\hat{\zeta}^{\dot\alpha A}\big)\,, \tag{26}$$

In terms of which the coupling to the vector completes the action (25) to

$$S(A_\mu, \Psi) = -\frac{1}{2}\text{tr}_k\,[A_\mu, A_\nu]^2 + \text{tr}_k\,\bar{\Psi}\Gamma_\mu\,[A_\mu, \Psi]\,. \tag{27}$$

In (26) we have used the decomposition of the fermionic collective coordinates

$$\begin{aligned}\mathcal{M}'^{A}_{\alpha} &= 4\xi^{A}_{\alpha}1_{[k]\times[k]} + 4\hat{a}'_{\alpha\dot\alpha}\bar\eta^{\dot\alpha A} + \hat{\mathcal{M}}'^{A}_{\alpha}\,,\\ \zeta^{\dot\alpha A} &= 4\bar\eta^{\dot\alpha A}1_{[k]\times[k]} + \hat\zeta^{\dot\alpha A}\,,\end{aligned} \tag{28}$$

where ξ^A_α and $\bar\eta^{\dot\alpha A}$ are the 4 supersymmetric and 4 superconformal fermion modes, respectively. In (27) the Γ_μ are a representation of the six-dimensional Clifford algebra.

Finally our effective measure for the k instantons has the form

$$\begin{aligned}\int d\mu^k_{\rm phys}\, e^{-S^k_{\rm inst}} \underset{N\to\infty}{=} & \frac{N^{1/2}g^4}{2^{9k^2/2-k/2+12}\,\pi^{5k^2/2+4}\,{\rm Vol}\,U(k)}k^{-1}e^{2\pi ik\tau}\times\\ &\times\int d\rho\, d^4X\, d\phi\, \rho^{-5}\, e^{4i\phi}\prod_{A=1,2}d^2\xi^A d^2\bar\eta^A\cdot\hat{\mathcal{Z}}^{(6)}_k\,,\end{aligned} \tag{29}$$

where $\hat{\mathcal{Z}}^{(6)}_k$ is the partition function of an $\mathcal{N}=(1,0)$ supersymmetric SU(k) gauge theory in six dimensions dimensionally reduced to zero dimensions:

$$\hat{\mathcal{Z}}^{(6)}_k = \int_{{\rm SU}(k)} d^6A\, d^8\Psi\, e^{-S(A_\mu,\Psi)}\,. \tag{30}$$

When integrating expressions which are independent of the SU(k) degrees-of-freedom, $\hat{\mathcal{Z}}^{(6)}_k$ is simply an overall constant factor that was evaluated in [15]. In our notation[3]

$$\hat{\mathcal{Z}}^{(6)}_k = \big(\sqrt{2\pi}\big)^{6(k^2-1)}\big(\sqrt{2}\big)^{(8-6)(k^2-1)}\cdot\frac{2^{k(k+1)/2}\pi^{(k-1)/2}}{2\sqrt{k}\prod_{i=1}^{k-1}i!}\cdot\frac{1}{k^2}\,. \tag{31}$$

In summary, the effective large-N collective coordinate measure has the following simple form:

$$\int d\mu^k_{\rm phys}\, e^{-S^k_{\rm inst}} \underset{N\to\infty}{=} \frac{\sqrt{N}g^4}{2^{17}\pi^{15/2}}\,k^{-7/2}e^{2\pi ik\tau}\int\frac{d^4X\,d\rho}{\rho^5}\,d\phi\, e^{4i\phi}\prod_{A=1,2}d^2\xi^A d^2\bar\eta^A\,. \tag{32}$$

This has a remarkable similarity to the form of the $\mathcal{N}=4$ measure (see [2, eq. (5.45)]). Apart from the differences in the overall numerical factors, the integral over S^5 is replaced by S^1 and the $\mathcal{N}=4$ measure involves, in addition, the factor $\sum_{d|k}d^{-2}$, the sum over the integer divisors of k. Notice that the $\sqrt{N}$ dependence and factor of $k^{-7/2}$ is the same in both cases. We can sum the instanton series in this case in terms of a generalized zeta-function:[4]

$$\sum_{k=1}^{\infty}k^{-7/2}e^{2\pi ik\tau} = e^{2\pi i\tau}\Phi\left(e^{2\pi i\tau},\frac{7}{2},1\right). \tag{33}$$

[3]We have written the result in a way which allows an easy comparison with [15]. The factors of $\sqrt{2\pi}$ and $\sqrt{2}$ arise, respectively, from the difference in the definition of the bosonic integrals and the normalization of the generators: we have ${\rm tr}_k\,T^rT^s=\delta^{rs}$ rather than $\frac{1}{2}\delta^{rs}$. The remaining factors are the result of [15]. We have also used ${\rm Vol}\,U(k)=2^k\pi^{k(k+1)/2}/\prod_{i=1}^{k-1}i!$.

[4]We thank Michael Nieto for pointing this out [16].

JHEP10(1999)019

The appearance of the phase in (32) implies a selection rule in order that correlation functions are non-vanishing. In fact this is exactly the selection rule (4) since, as is clear from the couplings of the fermion bilinears to χ in (10), each insertion of an adjoint fermion mode, other than the 8 exact modes, (numbering $n_a - 8$) implies an insertion of $e^{i\phi/2}$ and each insertion of a fundamental fermion mode (numbering n_f) implies an insertion of $e^{-i\phi/2}$. Using this selection rule we see that the simplest kind of non-vanishing correlation functions would involve operators that saturate the 8 supersymmetric and superconformal zero modes and also 8 fundamental fermion modes, i.e. $n_a = 8$ and $n_f = 8$.

In the $\mathcal{N} = 4$ theory the large-N instanton measure could be interpreted directly as the D-instanton measure for type IIB string theory on $AdS_5 \times S^5$. The natural question to ask is whether there is any similar interpretation for the measure (32) in terms of D-instantons in six dimensions. Remarkably there is such a relation arising from the type IIB superstring compactified on $K3$ [9]. D-instantons arise from wrapping the world-volume of the D-string in ten dimensions on an S^2 in $K3$ at a special point in the latter where the S^2 vanishes. More generally, Green and Gutperle [9] were able to deduce the normalization of the measure for D-instantons in various dimensions (4, 6 and 10) from the exact expressions for certain higher-derivative terms in the supergravity effective actions in those dimensions. For arbitrary dimension d, the leading order piece in the weak coupling expansion has the generic form:

$$(\alpha')^{-1}(S_k)^{a_d} e^{-2\pi(S_k - ik\chi)} \hat{\mathcal{Z}}_k^{(d)} \big(1 + \mathcal{O}(S_k^{-1})\big) , \tag{34}$$

where $S_k = ke^{-\phi}$ is the k-instanton action, where ϕ is the expectation value of the dilaton, the NS-NS scalar, and χ is the expectation value of the R-R scalar. In the above a_d is a constant that depends upon the dimension and $\hat{\mathcal{Z}}_k^{(d)}$ is the partition function of SU(k) minimally supersymmetric gauge theory in d-dimensions dimensionally reduced to zero dimensions, as in (30), but with a different normalization:

$$\hat{\mathcal{Z}}_k^{(10)} = \sum_{d|k} d^{-2} , \qquad \hat{\mathcal{Z}}_k^{(6)} = \hat{\mathcal{Z}}_k^{(4)} = k^{-2} . \tag{35}$$

Hence in six dimensions, where $a_6 = -3/2$, the leader order behaviour is

$$(\alpha')^{-1} e^{3\phi/2} k^{-7/2} \exp(2\pi ik(ie^{-\phi} + \chi)) . \tag{36}$$

With the usual identification of coupling constants of the Yang-Mills theory at large N and the string theory parameters; namely $g = \sqrt{4\pi e^{\phi}}$, $\theta = 2\pi\chi$ and $(\alpha')^{-1} \sim \sqrt{g^2 N}$, (36) goes like

$$\sqrt{N} g^4 k^{-7/2} e^{2\pi ik\tau} , \tag{37}$$

which matches the normalization of (32) exactly.

It remains to be seen whether the relation of the D-instanton measure of the type IIB string compactified on the special $K3$ and the large N multi-instanton measure in $\mathcal{N}=2$ SU(N) gauge theory with $2N$ fundamental hypermultiplets is a reflection of an AdS/CFT-type duality. However, we should make the following cautionary remark. It is believed that for a conformal gauge theory to admit a SUGRA dual in the large N limit the two coefficients of the four-dimensional anomaly, denoted as a and c in [17], have to be equal at leading order in $1/N$ [18, 19]. For the $\mathcal{N}=2$ conformal theory that we are considering this is not the case since $a=\frac{1}{24}(7N^2-5)$ and $c=\frac{1}{6}(2N^2-1)$ [20].[5] This implies that the dual, if it exists, is a string theory which does not have a SUGRA approximation.

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